On how Descartes changed the meaning of the Pythagorean Theorem

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Problem

$\square, \square = \square$ Euclid, *Elements*, I.47

$a^2 + b^2 = c^2$ $a, b, c \in \mathbb{R}$

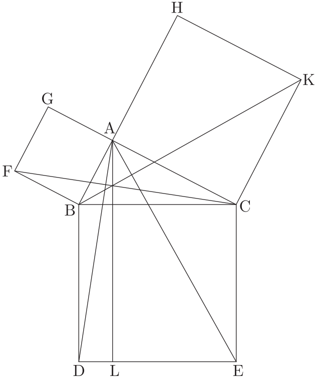
$\square, \square = \square$	Euclid's theory of equal figures
$a^2 + b^2 = c^2$	arithmetic of real numbers

The thesis

$\square, \square = \square$	Euclid's theory of equal figures
$a^2 + b^2 = c^2$	Descartes' arithmetic of line segments
$a^2 + b^2 = c^2$	arithmetic of real numbers

Elements, I.47

"In right-angled triangles the square on (ἀπό) the side subtending the right-angle is equal to the [sum of the] squares on (ἀπό) the sides surrounding the right-angle.



Let ABC be a right-angled triangle having the right-angle BAC. I say that the square on (ἀπό) BC is equal to the [sum of the] squares on (ἀπό) BA, AC".

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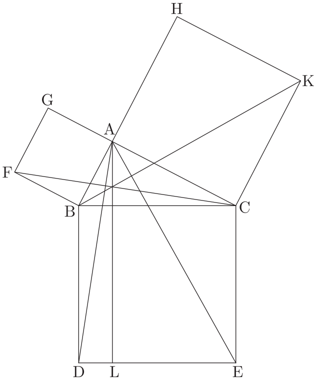

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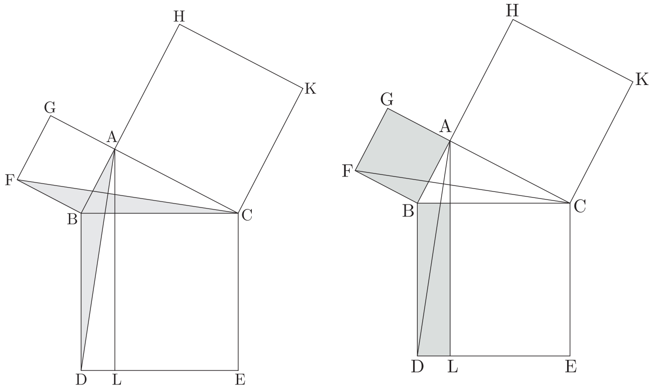
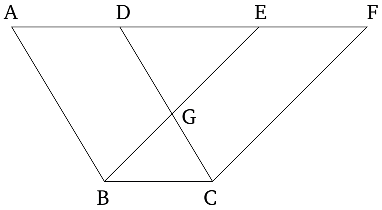
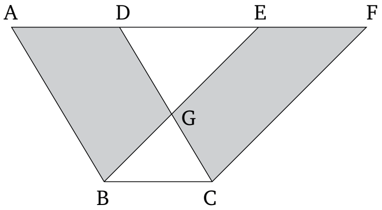
The proof of the proposition I.47



Euclid's theory of equal figures





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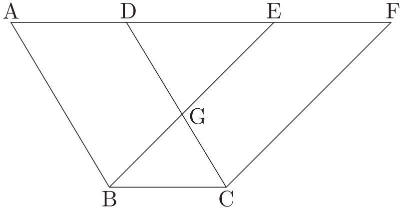
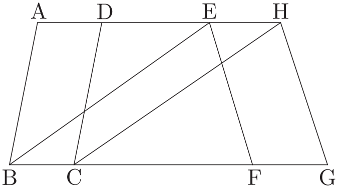
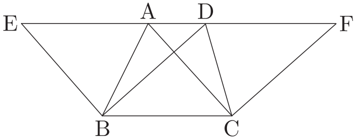
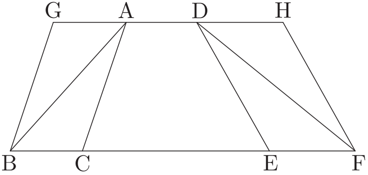
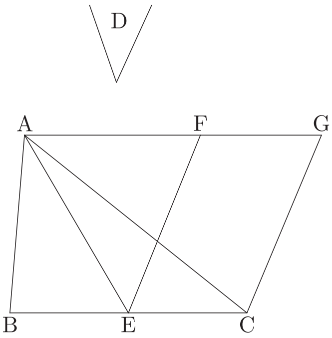


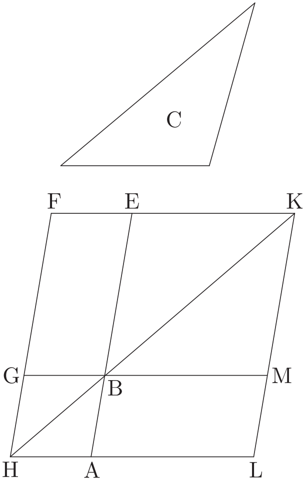




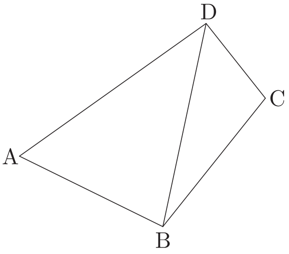


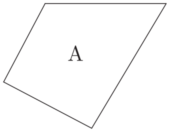
Euclid's theory of equal figures

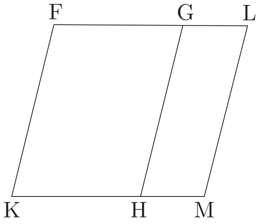


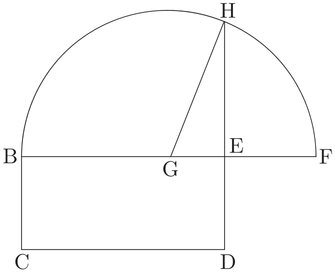


Euclid's theory of equal figures

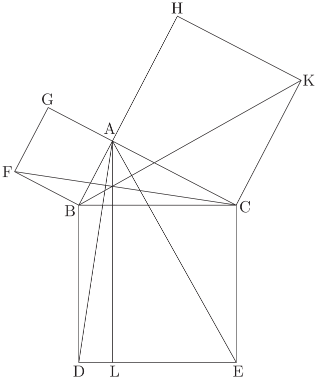








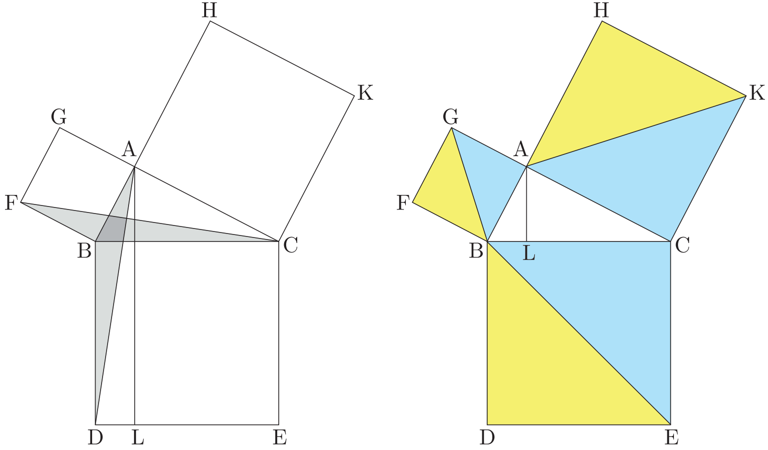
Elements, I.47



$\square, \square = \square$

Euclid's theory of equal figures

Pythagorean theorem via propositions I.47 and VI.31



V.24: "I say that the first and the fifth, added together (συντεθέν), AG, will also have the same ratio to ...".

$$a : c :: d : f, \; b : c :: e : f \Rightarrow (a + b) : c :: (d + e) : f$$

Problem

$\square, \square = \square$	Euclid's theory of equal figure
$\square + \square = \square$	Euclid's theory of similar figures
$a^2 + b^2 = c^2$	arithmetic of real numbers

The thesis

$\square, \square = \square$	Euclid's theory of equal figure
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$a^2 + b^2 = c^2$	Descartes' arithmetic of line segments
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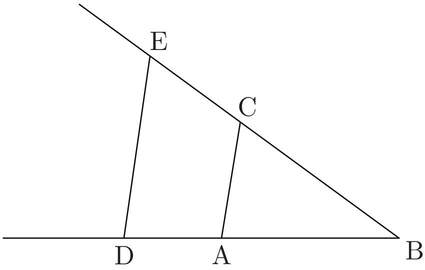
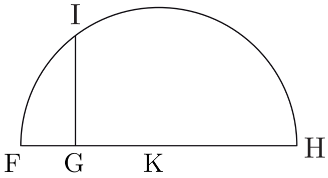
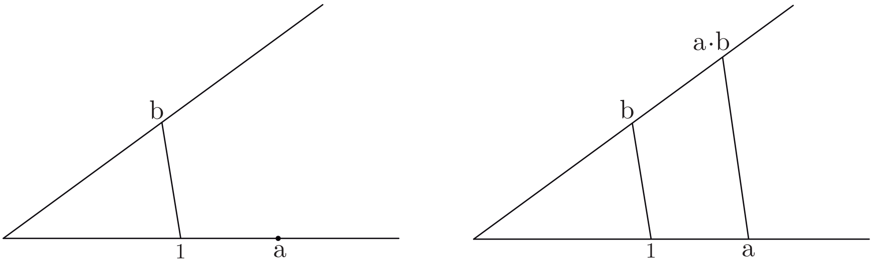
Descartes, *Geometry*, p. 298



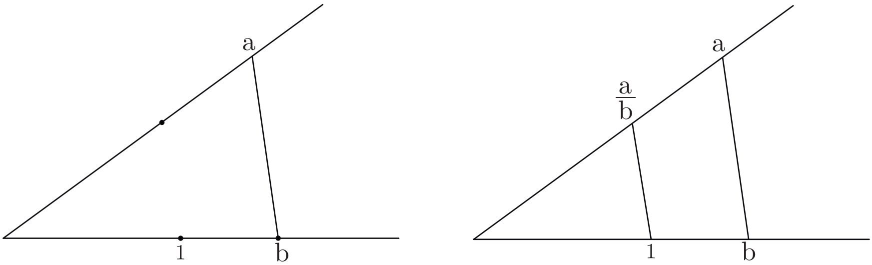


Arithmetic of line segments: Product



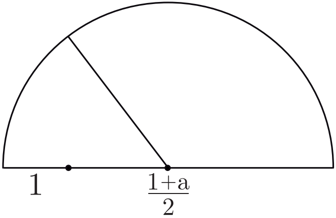
$$x = a \cdot b \ \ \textit{iff} \ \ 1 : b :: a : x.$$

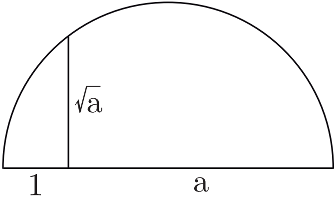
Arithmetic of line segments: Division



$$\color{red}{x = \frac{a}{b}} \quad \textit{iff} \quad \color{blue}{a : b :: x : 1}.$$

Arithmetic of line segments: Square Root





$x = \sqrt{a}$ *iff* $1 : x :: x : a$.

From Greek proportion to Descartes' arithmetic and backwards

$$a : b :: c : d \quad \Leftrightarrow \quad a \cdot d = c \cdot b$$
$$a : b :: c : d \quad \Leftrightarrow \quad a = \frac{c}{d} \cdot b$$

From Greek proportion to Descartes' arithmetic

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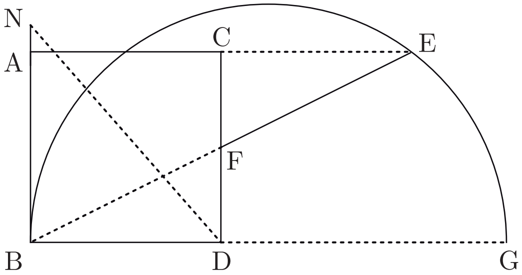


„For, putting a for BD or CD, and c for EF, and x for DF, we have CF $= a - x$, and since CF or $a - x$ is to FE or c, as FD or x is to BF, which is consequently $\frac{cx}{a-x}$".

$$CF : FE :: FD : BF$$

$$(a - x) : c :: x : BF \Rightarrow BF = \frac{cx}{a - x}$$

Ordered field of line segments

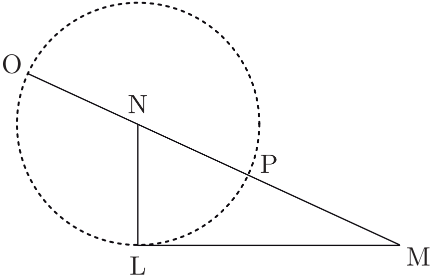
$$a : b :: c : d \quad \Rightarrow \quad a \cdot d = c \cdot b$$
$$a : b :: c : d \quad \Rightarrow \quad a = \frac{c}{d} \cdot b$$

$$\begin{aligned} a \cdot b &= b \cdot a \\ a \cdot (b \cdot c) &= (a \cdot b) \cdot c \\ a \cdot (b + c) &= a \cdot b + a \cdot c \\ a < b &\Rightarrow a \cdot c < b \cdot c \end{aligned}$$

$$a \cdot 1 = a/1 = a, \quad a/a = 1, \quad (1/a) \cdot a = 1$$
$$\frac{a}{b} \cdot c = \frac{a \cdot c}{b}, \quad \frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a \cdot b}$$
$$\sqrt{a} \cdot \sqrt{a} = a, \quad \sqrt{a^2} = a, \quad \sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

Pythagorean theorem in *Geometry*, Book I, p.302

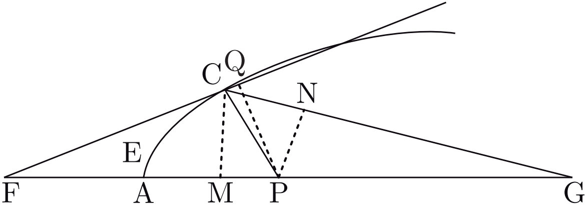
Solving the equation $z^2 = az + bb$



"I construct a right triangle NLM with one side LM equal to b, the square root of the known quantity bb, and the other side LN, equal to $\frac{1}{2}a$, that is, the half of other know quantity, which was multiplied by z, which I supposed to be the unknown line. Then prolonging MN the base of this triangle to O, so that NO is equal to NL, the whole line OM is the required line z. It is expressed in the following way: $z = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$".

Pythagorean theorem in *Geometry*, Book II, p. 342

Finding a tangent to an algebraic curve



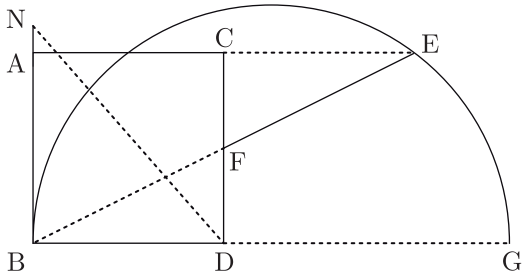
"[...] making MA or CB $= y$, and CM or BA $= x$, I have an equation expressing the relation between x and y. Then making PC $= s$, and PA $= v$, or PM $= v - y$, and due to the right triangle PMC, we see that ss, the square of the base (*qui est le carreé de la baze*), is equal to $xx + vv - 2vy + yy$, the squares of two sides (*qui sont les carées des deux côtés*); that is to say

$$x = \sqrt{ss - vv + 2vy - yy}\text{''}.$$

$$ss = x^2 + (v - y)^2$$

Pythagorean theorem in *Geometry*, Book III, p. 387

Solving a fourth degree equation



“For, putting a for BD or CD, and c for EF, and x for DF, we have CF $= a - x$, and since CF or $a - x$ is to FE or c, as FD or x is to BF, which is consequently $\frac{cx}{a-x}$. Now, since in the right triangle BDF one side is x, and other is a, their squares, which are (*leurs carrés, qui sont*) $xx + aa$, are equal to the square of the base, which is $\frac{ccxx}{xx-2ax+aa}$”.

$$x^2 + a^2 = \frac{(cx)^2}{(a-x)^2}$$

Book II: “[the sum of] the squares of two sides (*qui sont les carées des deux côtés*); that is to say $x^2 + (v - y)^2$

Book III: “[the sum of] their squares, which are (*leurs carrés, qui sont*) $xx + aa$

Linguistic layer of *La Géométrie*

square on the line *versus* square of the line

"we see that ss, the square of the base (*qui est le carreé de la baze*), is equal to $xx + vv - 2vy + yy$, the squares of two sides (*qui sont les carées des deux côtés*)", Book II

"since in the right triangle BDF one side is x and other is a, their squares, which are (*leurs carrés, qui sont*) $xx + aa$, are equal to the square of the base, which is $\frac{ccxx}{xx-2ax+aa}$", Book III

Descartes' translation of Pappus. *Geometry*, p. 305

square on the line *versus* square of the line

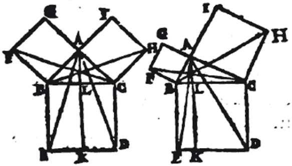
Pappus: "if there be given a ratio (λόγος) of the rectangle contained by the two lines so drawn to the square on the third"

Descartes: "let there be a proportion (*proportio*) of the rectangle contained by the two lines so drawn to the square of the third (*ad quadratum reliquæ*)"

Clavius, *Euclidis Elementorum*, p. 229

THEOR. 33. PROPOS. 47. 46.

IN rectangulis triangulis, quadratū, quod a latere rectum angulum ſubtendente deſcribitur, æquale eſt eis, quæ a lateribus rectum angulum continentibus deſcribuntur, quadratis.



IN triãgulo ABC, angulus BAC, ſit rectus,[a] deſcribanturque ſuper AB, AC, BC, quadrata ABFG, ACHI, BCDE. Dico quadratum BCDE, deſcriptum ſuper latus BC, quod angulo recto opponitur, æquale eſſe duobus quadratis ABFG, ACHI, quæ ſuper alia duo latera ſunt deſcripta, ſiue hæc duo latera æqualia ſint, ſiue inæqualia. Ducatur enim recta AK,[b] parallela ipſi BE, vel ipſi CD, ſecans BC, in L, & iungantur rectæ AD, AE, CF, BH. Et quia duo anguli BAC, BAG, ſunt recti,[c] erũt rectæ GA, AC, vna linea recta; eodemque modo IA, AB, vna recta linea erunt. Rurſus quia anguli ABF,

[a] 46. primi.

[b] 31. primi.

[c] 14. primi.

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