Geometric Hydrodynamics

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V. Arnold observed in his seminal paper that solutions of the Euler equations for ideal fluid motion can be viewed as geodesics of a certain right-invariant metric on the group of volume-preserving diffeomorphisms (known as volumorphisms), $D_\mu(M)$. In essence, this approach showcases the natural framework in which to tackle this infamous Cauchy problem from the so-called Lagrangian viewpoint. In their celebrated paper Ebin and Marsden provided the formulation of the above in the H^s Sobolev setting. Here they proved that the space of H^s volumorphisms can be given the structure of a smooth, infinite dimensional Hilbert manifold. They illustrated that, when equipped with a right-invariant L^2 metric, the geodesic equation on this manifold is a smooth ordinary differential equation. They then applied the classic iteration method of Picard to obtain existence, uniqueness and smooth dependence on initial conditions. In particular, the last property allows one to define a smooth exponential map on $D^s_\mu(M)$ in analogy with the classical construction in finite dimensional Riemannian geometry. Hence, the work of Arnold, Ebin and Marsden allows one to explore the problem of ideal fluid motion armed with tools from Riemannian geometry.